

# Econometrics

## Practical Session 18

### Nonstationarity: Deterministic and Stochastic Trends

---

Ricardo Gouveia-Mendes

rgouveiamendes@ucp.pt

Spring 2025-26

Católica-Lisbon School of Business and Economics



CATÓLICA  
LISBON  
BUSINESS & ECONOMICS

# Theoretical Wrap-up

---

# What Is a Trend?

- A **trend** is a persistent, long-term movement in a time series
- Two fundamentally different types:

## Deterministic trend

$$y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a fixed function of time
- $\varepsilon_t$  is stationary
- Shocks are **temporary**:  $y_t$  reverts to  $\alpha + \beta t$  after a shock

## Stochastic trend (random walk)

$$y_t = y_{t-1} + \varepsilon_t$$

- Trend is random
- Shocks are **permanent**: each  $\varepsilon_t$  shifts the level forever → no fixed mean to revert to

- Both introduce **non-stationarity** → each has a particular remedy

# Deterministic vs Stochastic Trends: Moments

## Deterministic trend

$$y_t = a + \beta t + \varepsilon_t$$

- $\mathbb{E}[y_t] = a + \beta t$
- $\text{var}(y_t) = \sigma_u^2$  (stable)

## Random walk

$$y_t = y_{t-1} + \varepsilon_t$$

- $\mathbb{E}[y_t] = y_0$
- $\text{var}(y_t) = t\sigma^2 \rightarrow +\infty$

## RW with drift

$$y_t = \mu + y_{t-1} + \varepsilon_t$$

- $\mathbb{E}[y_t] = y_0 + \mu t$
- $\text{var}(y_t) = t\sigma^2 \rightarrow +\infty$

- **Unit root** ( $\beta_1 = 1$  in AR(1))  $\Leftrightarrow$  stochastic trend; shocks accumulate permanently
- **No unit root** ( $|\beta_1| < 1$ )  $\Leftrightarrow$  deterministic trend or stationary; history fades

# Problems Caused by Stochastic Trends

- 1. Biased AR coefficients:** OLS estimates of  $\beta_1$  in an AR(1) are biased toward 1  
→ forecasts are unreliable
- 2. Invalid inference:**  $t$ -statistics do not follow the standard normal distribution even in large samples
- 3. Spurious regression:** regress two **independent** random walks on each other:
  - OLS gives highly significant coefficients and  $R^2$  close to 1
  - The result is pure noise → “trending together” in a plot is **not** evidence of a relationship
- **Rule:** check for stationarity before any regression or AR model

# The Dickey-Fuller and Augmented Dickey-Fuller Tests

- Consider that:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t \Leftrightarrow \Delta y_t = \beta_0 + \underbrace{(\beta_1 - 1)}_{\delta} y_{t-1} + \varepsilon_t$$

- **Null hypothesis:**  $H_0 : \delta = 0$  (i.e.  $\beta_1 = 1$ , unit root)  
**Alternative:**  $H_1 : \delta < 0$  (i.e.  $\beta_1 < 1$ , mean-reverting) → one-sided
- **Critical values follow a non-standard distribution** → Dickey-Fuller table
- **Augmented DF (ADF):** add **lagged differences** to absorb serial correlation in  $\varepsilon_t$ , **choosing  $p$  by BIC or by the residuals ACF** (so that they become white noise):

$$\Delta y_t = \beta_0 + \delta y_{t-1} + \gamma_1 \Delta y_{t-1} + \dots + \gamma_{p-1} \Delta y_{t-p+1} + \varepsilon_t$$

# ADF: Intercept Only vs. Intercept + Trend

Two test regressions, with **different critical values**:

## (a) Intercept only

$$\Delta y_t = \beta_0 + \delta y_{t-1} + \dots + \varepsilon_t$$

- Alternative:  $y_t$  stationary around a **constant**
- Use when the series has **no visible long-run trend**

## (b) Intercept + time trend

$$\Delta y_t = \beta_0 + \mu t + \delta y_{t-1} + \dots + \varepsilon_t$$

- Alternative:  $y_t$  stationary around a **linear trend**
- Use when the series **trends upward or downward** over time

- **Rule of thumb:** look at the data first → if  $y_t$  trends, include  $t$   
→ Misspecification matters

# Dickey-Fuller Critical Values

Significance	(a) Intercept only	(b) Intercept + trend
1%	-3.43	-3.96
5%	-2.86	-3.41
10%	-2.57	-3.12

Asymptotic critical values. Reject  $H_0$  if  $\tau <$  critical value.  
Source: MacKinnon (1991), Stock & Watson Table 15.4.

# Achieving Stationarity: First Differencing

- If  $y_t$  has a unit root, its **first difference**  $\Delta y_t = y_t - y_{t-1}$  is typically stationary
- **Integrated of order 1** ( $I(1)$ ):  $y_t$  nonstationary,  $\Delta y_t$  stationary
  - GDP **level** is  $I(1)$ ; GDP **growth** =  $400 \times \Delta \ln \text{GDP}_t$  is  $I(0)$
  - CPI **level** is  $I(1)$ ; inflation =  $400 \times \Delta \ln \text{CPI}_t$  is  $I(0)$
- **Practical advice:** if  $y_t$  has a stochastic trend, model  $\Delta y_t \rightarrow$  not  $y_t$  in levels
- If  $y_t$  has a **deterministic** trend, subtract the trend (detrend) rather than differencing
- **Do not difference unnecessarily:** over-differencing induces spurious MA structure and destroys long-run information

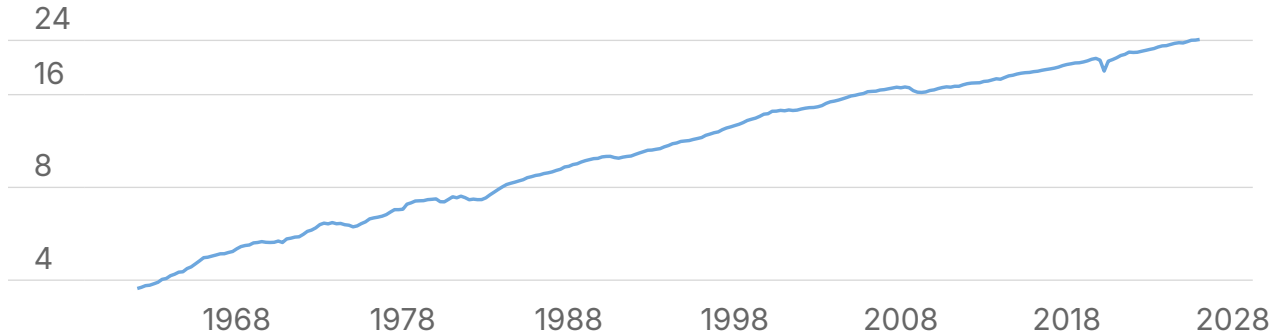
# Data Overview

---

# Real GDP and GDP Growth

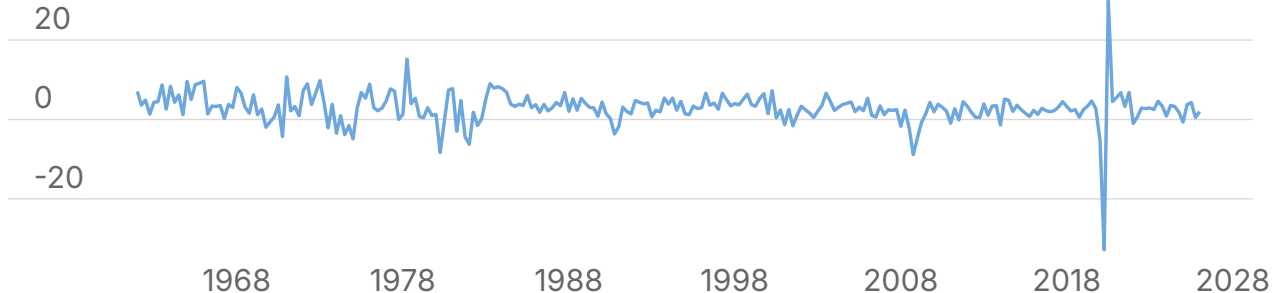
## US Real GDP Level

(trillions, log scale)



## US Real GDP Growth

(%, annualized)



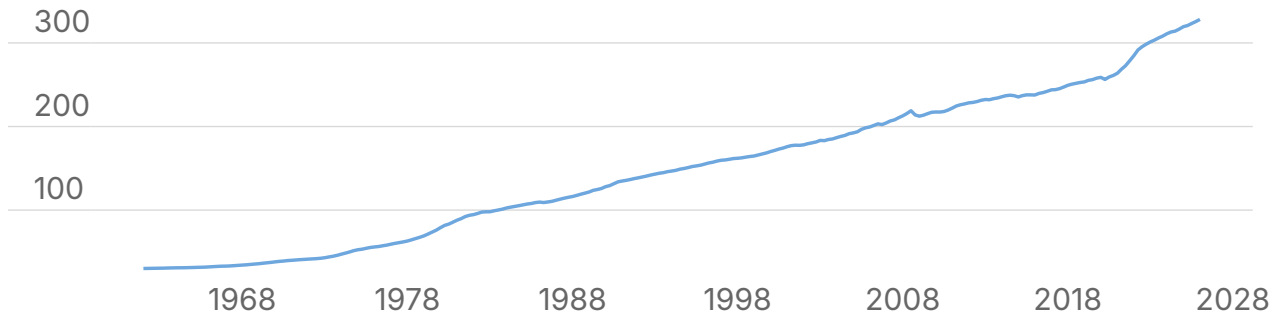
Source: FRED, series GDBP1

- **Real GDP level** (top): billions of chained 2017 USD, log scale
- **GDP growth** (bottom): annualised quarterly growth rate ( $400 \times \Delta \ln \text{GDP}_t$ )
- Quarterly data, 1962–2025

# CPI and Inflation

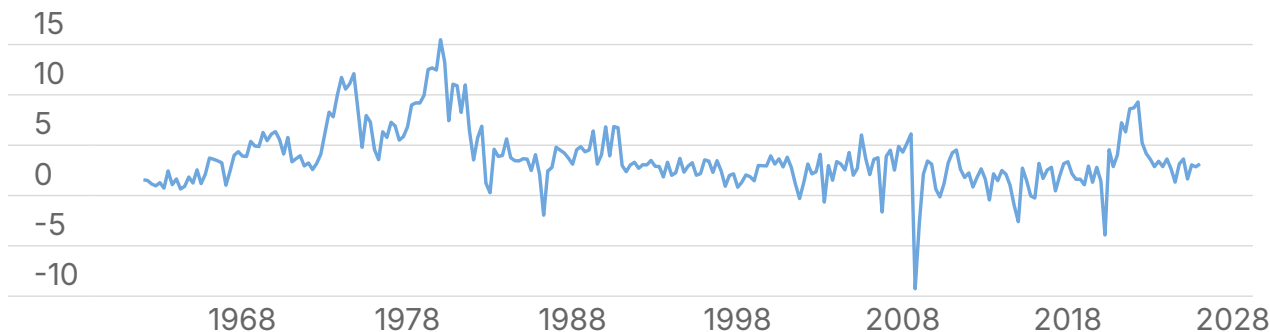
## CPI Level

(index, 1982–84 = 100)



## Inflation Rate

(%, annualized)

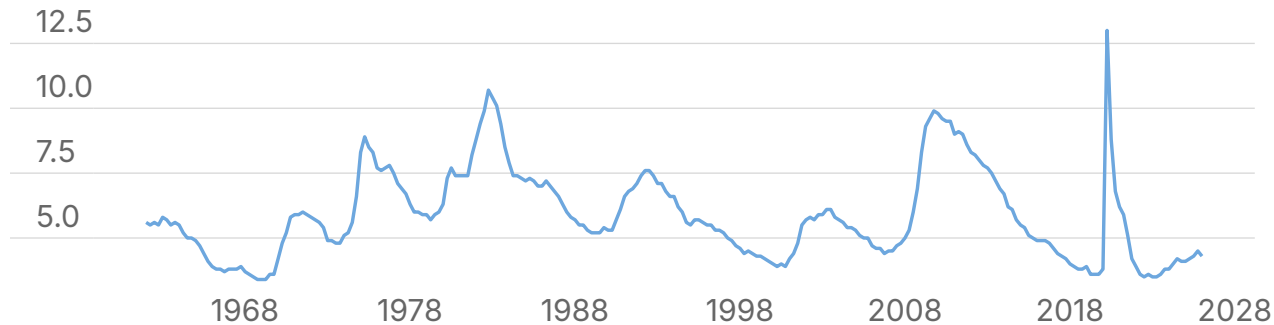


Source: EPRU, series CPILICSI

- **CPI level** (top):  
Consumer Price Index,  
1982–84 = 100
- **Inflation** (bottom):  
annualised quarterly  
inflation rate ( $400 \times$   
 $\Delta \ln \text{CPI}_t$ )
- Quarterly data, 1962–  
2025

# Unemployment Rate

## Unemployment Rate (%)



## Change in Unemployment Rate (pp)

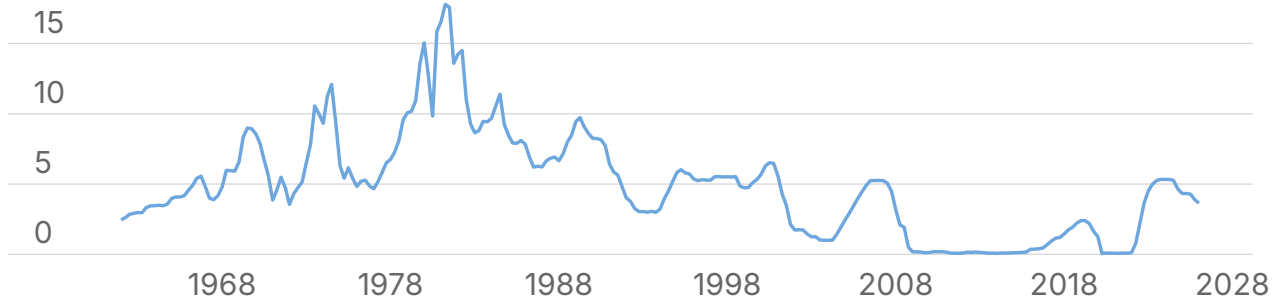


Source: EDED, series UNDATE

- **Unemployment rate**  
(top): civilian  
unemployment rate (%)
- **Change in  
unemployment**  
(bottom): first difference  
( $\Delta \text{une}_t$ )
- Quarterly data, 1962–  
2025

# Federal Funds Rate

## Federal Funds Rate (%)



## Change in Federal Funds Rate (pp)



Source: FRED, series FFRBLINDS

- **Federal Funds Rate** (top): the overnight interbank lending rate set by the Fed
- **Change in FFR** (bottom): first difference ( $\Delta FFR_t$ )
- Quarterly data, 1962–2025

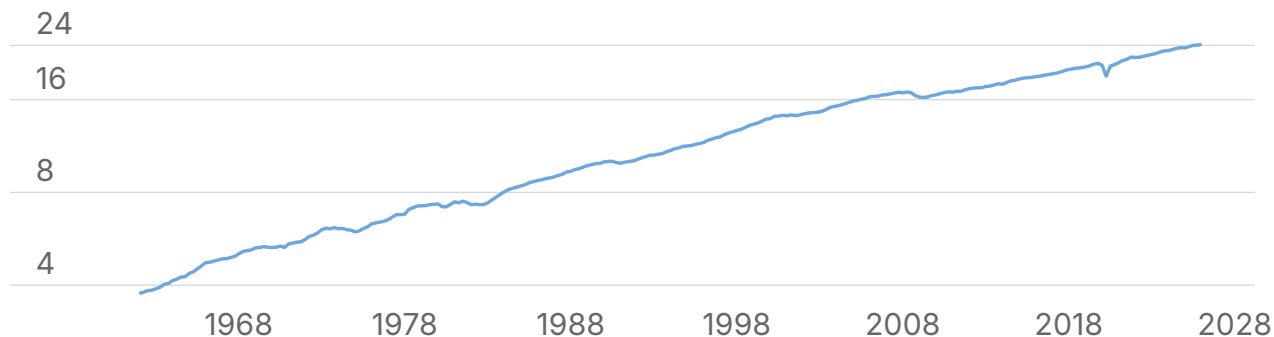
# Exercises

---

# Exercise 1 | GDP Level vs. GDP Growth

## US Real GDP Level

(trillions, log scale)



## US Real GDP Growth

(%, annualized)



Source: FRED, series GDP1

a) The GDP level and GDP growth rate are plotted on the left. Which looks stationary? What visual features of each series inform your assessment?

# Exercise 1 | GDP Level vs. GDP Growth

## KEY TAKEAWAYS

### GDP level: nonstationary

- Rises continuously
- No tendency to revert to a mean
- Variance clearly grows over time

- **Lesson:** always transform levels to growth rates (or log-differences) for trending variables before building AR/ADL models

### GDP growth: stationary

- No apparent direction
- Fluctuates around a roughly stable mean (about 3%)
- Roughly constant spread

# Exercise 1 | GDP Level vs. GDP Growth

**b)** Augmented Dickey-Fuller tests are run on both series:

ADF test on GDP level:

Dickey-Fuller =  $-0.63$ , Lag order = 6, p-value =  $0.98$   
=> Fail to reject  $H_0$ : unit root present.

ADF test on GDP growth:

Dickey-Fuller =  $-6.28$ , Lag order = 6, p-value =  $0.01$   
=> Reject  $H_0$ : unit root absent.

Are these results consistent with the visual assessment in a)?

# Exercise 1 | GDP Level vs. GDP Growth

## KEY TAKEAWAYS

### GDP level: $p = 0.98$

- High  $p$ -value
- Cannot reject unit root
- Nonstationary → consistent with the visual

- The ADF confirms what we saw: the level wanders without bound; the growth rate fluctuates around a stable mean

### GDP growth: $p = 0.01$

- Low  $p$ -value
- Reject unit root
- Strong evidence of stationarity → consistent with the visual

# Exercise 1 | GDP Level vs. GDP Growth

c) The ADF in b) used the **intercept + trend** specification. Compare results from both specifications on GDP level:

ADF (intercept only):

$\tau = 2.67$ , 5% c.v. =  $-2.87$   $\Rightarrow$  Fail to reject  $H_0$ .

ADF (intercept+trend):

$\tau = -0.87$ , 5% c.v. =  $-3.42$   $\Rightarrow$  Fail to reject  $H_0$ .

$\Rightarrow$  Both fail to reject  $H_0$ : unit root present under both specifications.

The intercept-only  $\tau$  is positive, while the other is negative. What does this mean? Which specification is appropriate, and why?

# Exercise 1 | GDP Level vs. GDP Growth

## KEY TAKEAWAYS

- **A positive tau means  $\hat{\delta} > 0$ :** without controlling for the trend, the series looks **explosive**, not mean-reverting → misspecification artefact, not a real economic finding
- **Intercept+trend is appropriate:** GDP level has a clear persistent upward linear trend
- **Both specifications fail to reject:** GDP level is  $I(1)$  regardless → but the intercept-only result is meaningless for a trending series
- **Lesson:** for a strongly trending series, always use intercept + trend; the drift specification can produce uninterpretable (even positive) tau statistics

## Exercise 2 | Spurious Regression

a) A researcher regresses the GDP **level** on the CPI **level**:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	1995.6	86.8	23.0	< 0.001 ***
cpi	68.5	0.5	133.7	< 0.001 ***

$R^2 = 0.986$

The  $R^2$  is 0.99 and the  $t$ -statistic is 133.7. Is this evidence of a genuine economic relationship between the price level and real GDP?

# Exercise 2 | Spurious Regression

## KEY TAKEAWAYS

- **No** → **this is a spurious regression**: both series are trending upward over time for completely independent reasons
- High  $R^2$  and large  $t$ -statistics arise mechanically whenever two  $I(1)$  series share a common upward trend
- The coefficient (68.5) suggests that a 1-unit rise in the CPI is “associated” with \$68.5bn more GDP → economically nonsensical
- **Lesson**: if you regress any two trending series, you will get  $R^2 > 0.9$  regardless of whether they are related

## Exercise 2 | Spurious Regression

b) The same researcher redoes the analysis using GDP **growth** and **inflation**:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	2.712	0.420	6.45	< 0.001 ***
inf	0.058	0.088	0.66	0.509

$R^2 = 0.002$

How do  $R^2$  and the  $p$ -value change relative to a)? What is the correct conclusion about the growth–inflation relationship?

# Exercise 2 | Spurious Regression

## KEY TAKEAWAYS

- $R^2$  collapses from 0.99 to 0.002 → **inflation explains essentially none of the variation in GDP growth**
- The coefficient is not significant ( $p = 0.51$ )
- If you were to use pre-2018 data, the sign is would be negative → another sign of a spurious relationship
- **Correct conclusion:** there is no simple contemporaneous linear relationship between GDP growth and inflation in this sample; the spurious level regression was misleading in every dimension
- **Lesson:** the apparent relationship between the levels was a statistical artefact → both series were trending upward together, not because they are related

## Exercise 2 | Spurious Regression

c) What does the spurious regression example tell us about how to read empirical papers that use macroeconomic data in levels?

## Exercise 2 | Spurious Regression

c) What does the spurious regression example tell us about how to read empirical papers that use macroeconomic data in levels?

### KEY TAKEAWAYS

- Regressions in levels between trending variables **always** show high  $R^2$  and significant coefficients, but no relationship
- **Red flag:** papers that regress GDP on money supply, population, or any other trending variable **in levels** without testing for cointegration or detrending — even if the economic relationship is real, the regression alone cannot distinguish a genuine signal from a spurious one
- **The standard:** applied macro papers either (i) work in growth rates/first differences, (ii) test for cointegration before using levels, or (iii) include a time trend to absorb common growth
- **Lesson for your project:** never use electricity consumption **in levels** without checking for stationarity

# Exercise 3 | Testing Inflation: How Many Differences?

a) ADF tests on inflation and on the **change** in inflation ( $\Delta \text{inf}_t$ ):

ADF on inflation (intercept only):

$\tau = -4.77$ , 5% c.v. =  $-2.87$   $\Rightarrow$  Reject  $H_0$ : stationary.

ADF on  $\Delta$ inflation (intercept only):

$\tau = -6.45$ , 5% c.v. =  $-2.87$   $\Rightarrow$  Reject  $H_0$ : stationary.

The ADF rejects the unit root for inflation. Yet the CPI plot shows inflation trending strongly through the 1970s. Should we trust the test and treat inflation as  $I(0)$ ?

# Exercise 3 | Testing Inflation: How Many Differences?

## KEY TAKEAWAYS

- **Specification choice:** inflation has no visible linear trend — it rises, peaks around 1980, then falls and stabilises; therefore the intercept-only spec is appropriate; adding a time trend would be misspecified and is not shown
- **The rejection likely reflects a structural break (Volcker disinflation, 1979–82):** inflation's mean shifted permanently downward; the ADF “sees” mean reversion around the pre- and post-break means and interprets it as stationarity
- **Visual inspection contradicts the test:** a genuinely stationary series fluctuates around a stable mean; inflation trended upward for 20 years and then collapsed → not consistent with stationarity
- **Practical conclusion:** inflation is borderline; better to use  $\Delta \text{inf}_t$  because it is unambiguously  $I(0)$ , robust to structural breaks, and standard in macro forecasting

## Exercise 3 | Testing Inflation: How Many Differences?

**b)** Suppose you want to forecast inflation using lagged unemployment as a predictor. Given the uncertainty about inflation's integration order established in a), what is the safest modelling strategy and why?

# Exercise 3 | Testing Inflation: How Many Differences?

## KEY TAKEAWAYS

- **The uncertainty is the problem:** the ADF test in a) says inflation is  $I(0)$ , but we know that test is unreliable when there is a structural break; we cannot rule out that inflation is genuinely  $I(1)$  or borderline; if it is, and unemployment is also non-stationary, OLS on levels risks spurious inference
- **Is unemployment stationary?** ADF tests on unemployment typically fail to reject the unit root — yet unemployment is bounded and mean-reverts to the natural rate, so it is probably  $I(0)$  with a slowly shifting mean; its integration order is debated, which is itself a reason for caution
- **The fix:** use  $\Delta \text{inf}_t$  as the dependent variable; it is unambiguously  $I(0)$  regardless of whether  $\text{inf}_t$  is truly  $I(1)$ , borderline, or  $I(0)$  with breaks — so OLS inference is valid without having to resolve the debate
- **Interpretation:** the regression now asks how unemployment predicts the **change** in inflation, not its level; this is also the empirical standard in Phillips-curve forecasting (Stock and Watson)

## Exercise 4 | Classifying Series by Integration Order

ADF tests (both specifications) are run on five US macroeconomic series (quarterly, 1962–2025), which are shown in the Data Overview above. Results are in the table below:

Series	tau (intercept only)	tau (intercept+trend)	Verdict
Real GDP (level)	2.67	-0.87	$I(1)$
GDP Growth	-10.09	-10.35	$I(0)$
CPI (level)	3.21	-0.72	$I(1)$
Inflation	-4.77	-5.16	$I(0)$
Fed Funds Rate	-2.55	-3.28	$I(1)$

# Exercise 4 | Classifying Series by Integration Order

a) For each series, explain **why** the tau statistics and specification choice lead to that verdict.

# Exercise 4 | Classifying Series by Integration Order

a) For each series, explain **why** the tau statistics and specification choice lead to that verdict.

## KEY TAKEAWAYS

- **Real GDP (level):** tau(drift) = 2.67 is **positive** → drift spec is misspecified for a trending series; tau(trend) = -0.87 fails to reject; verdict  $I(1)$ ; use intercept+trend
- **GDP Growth:** both tau statistics around -10; strongly reject at 1%; no trend needed → verdict  $I(0)$
- **CPI (level):** tau(drift) = 3.21 also positive (same story as GDP level); tau(trend) = -0.72 far from rejecting; verdict  $I(1)$ ; see 4c on what positive drift tau values reveal
- **Inflation:** both specs reject strongly; verdict  $I(0)$ , but see Exercise 3 for why the rejection may be misleading
- **Fed Funds Rate:** both fail to reject; verdict  $I(1)$ ; no clear linear trend, so intercept only is more appropriate

## Exercise 4 | Classifying Series by Integration Order

**b)** For each  $I(1)$  series, what is the standard transformation to achieve stationarity? Why is that particular transformation used rather than simply subtracting the sample mean?

# Exercise 4 | Classifying Series by Integration Order

**b)** For each  $I(1)$  series, what is the standard transformation to achieve stationarity? Why is that particular transformation used rather than simply subtracting the sample mean?

## KEY TAKEAWAYS

- **Real GDP:** growth rate =  $400 \times \Delta \ln \text{GDP}_t$  → log-differencing removes the stochastic trend and produces a meaningful economic variable (annualised %)
- **CPI:** inflation =  $400 \times \Delta \ln \text{CPI}_t$  → same reasoning; directly interpretable
- **Fed Funds Rate:** change in rate =  $\Delta \text{FFR}_t$  → differences remove persistence while keeping the units (percentage points)
- **Why not subtract the mean?** Subtracting the sample mean removes the **unconditional** mean but does nothing about a stochastic trend → the variance still grows, and the series still wanders
- **Lesson:** stationarity is about stabilising both the mean **and** the variance through time

# Exercise 4 | Classifying Series by Integration Order

c) Both Real GDP level and CPI level have **positive** tau statistics under the drift specification ( $\tau = 2.67$  and  $\tau = 3.21$  respectively). What does a positive tau mean, and why does this arise specifically for these two series?

# Exercise 4 | Classifying Series by Integration Order

## KEY TAKEAWAYS

- **What a positive tau means:** a positive tau corresponds to  $\hat{\delta} > 0$  → the series looks explosive rather than mean-reverting
- **This is a misspecification artefact:** both GDP level and CPI level grow persistently over time
- **Rule:** whenever you see a positive tau for an economic level series, the drift specification is misspecified → switch to intercept + trend
- **Under the trend specification:** both taus are negative and in the expected  $I(1)$  range → fail to reject
- **Lesson:** always visually inspect the series before choosing a specification; a trending series requires intercept + trend

# Discussion

1. The ADF test has  $H_0$ : unit root. Why does failing to reject  $H_0$  not prove that the series has a unit root? When should you be especially skeptical of a non-rejection?

1. The ADF test has  $H_0$ : unit root. Why does failing to reject  $H_0$  not prove that the series has a unit root? When should you be especially skeptical of a non-rejection?

## KEY TAKEAWAYS

- Failing to reject  $H_0$  only means we lack sufficient evidence against the unit root → it is not proof that a unit root exists
- The ADF has **low power** against near-unit-root alternatives (e.g.  $\beta_1 = 0.97$ ): a highly persistent but stationary series
- Be especially skeptical when: (i) the sample is short, (ii) the series is visually mean-reverting even if slowly, or (iii) the  $p$ -value is in the 0.10–0.50 range (ambiguous, not decisive)
- **Practical implication:** treat a non-rejection as “we cannot reliably distinguish this from a unit root” rather than “this series is definitely nonstationary” → when in doubt, difference

2. Looking ahead to the research project: how would you check whether the Portuguese electricity consumption series is stationary?

## 2. Looking ahead to the research project: how would you check whether the Portuguese electricity consumption series is stationary?

### KEY TAKEAWAYS

- **Step 1:** plot the series → does it trend? Does variance grow over time? Are there seasonal patterns that look like they compound (multiplicative seasonality)?
- **Step 2:** run an ADF test on the raw series — choose the intercept+trend specification if the level is visibly trending. If you fail to reject the unit root, first-difference it
- **Step 3:** electricity has strong weekly seasonality (same hour same day of week). The relevant stationary transform may be the **seasonal difference**:  $\Delta_7 y_t = y_t - y_{t-7}$
- **Step 4:** re-test after differencing. Use the stationary series for your AR/ADL model
- **Key point:** you want the residuals of your forecasting model to look like white noise → if the ACF of residuals still shows strong autocorrelation, the series is not yet fully stationary